

On the Stochastic Dynamics of Disordered Spin Models

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Received April 14 2003

In this article we discuss several aspects of the stochastic dynamics of spin models. The paper has two independent parts. Firstly, we explore a few properties of the multi-point correlations and responses of generic systems evolving in equilibrium with a thermal bath. We propose a fluctuation principle that allows us to derive fluctuation–dissipation relations for many-time correlations and linear responses. We also speculate on how these features will be modified in systems evolving slowly out of equilibrium, such as finite-dimensional or dilute spin-glasses. Secondly, we present a formalism that allows one to derive a series of approximated equations that determine the dynamics of disordered spin models on random (hyper) graphs.

KEY WORDS: Nonequilibrium dynamics; dilute disordered spin models; fluctuation formulae.

1. INTRODUCTION

There are several motivations to revisit the stochastic dynamics of spin models with and without disorder. Starting from a random initial condition, at low enough temperature, these models usually have a very slow evolution with several aspects in common with the one of real glassy systems. For instance, during coarsening the global auto-correlation functions of ferromagnetic Ising models on finite-dimensional lattices⁽¹⁾ age in a rather similar way to the one observed in molecular dynamic simulations of

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Lennard-Jones mixtures,⁽²⁾ a typical glass former. The asymptotic linear response⁽³⁾ to an external perturbation of these non-frustrated and non-disordered models is, however, different from the one measured in glasses. The non-trivial slow response observed numerically⁽²⁾ and experimentally⁽⁴⁾ is captured by modified spin models in which frustration and/or disorder are added.⁽⁵⁾ A standard example is the fully-connected disordered p spin model in which all p -uplets of spins interact via random exchanges. The statics⁽⁶⁾ and Langevin dynamics⁽⁷⁾ of this mean-field model have been solved analytically in great detail, and there is growing consensus that this and related models yield a mean-field description of the structural glass transition and glassy dynamics.

More recently, the interest in studying similar disordered spin models on random graphs and random hyper-graphs (as opposed to the complete graph or hyper-graph) has grown. The reasons for this are multiple.

From the glassy point of view, the dilute disordered models “approach” the finite-dimensional actual problem that one would like to understand while still being mean-field. Due to the dilute nature of the interactions not all fluctuations are suppressed in the thermodynamic limit. In more technical terms, when the number of spins in the system diverges, the disorder-averaged dynamics is not completely described by the two-time global auto-correlation and linear response but all kinds of many-time functions carry non-trivial information about the dynamic behavior of these systems. Since this will also happen in finite-dimensional problems one might expect that the knowledge on the behavior of these higher-order correlations in the dilute problem may be of help.

Dilute disordered spin models are highly non-trivial even in their paramagnetic high-temperature phase. As shown by Bray,⁽⁸⁾ these models have a Griffiths phase “generated” by the fluctuations in the connectivities of the vertices of the random (hyper) graph. A complete solution to the dynamics of one such model might help in understanding the nature and properties of Griffiths phases. In particular, the existence or not of a Griffiths phase in quantum disordered finite-dimensional systems in contact with a quantum environment has been the subject of intense debate recently.⁽⁹⁾ The analytic solution of a quantum model even if defined on a random graph might shed light onto this discussion.

Dilute disordered spin models also yield a representation of several problems of great interest in computer science.⁽¹⁰⁾ For instance, the dilute p spin disordered ferromagnetic model represents the so-called XOR-sat problem⁽¹¹⁾ and variations on the p spin disordered spin-glass model describe the k -sat optimization problem.^(12,13) These models are usually attacked with numerical algorithms that do not correspond to a physical dynamics like the Glauber or Langevin ones.⁽¹⁴⁾ Having said this, it would

be interesting and useful, also for these computational problems, to understand their evolution under physical dynamics.

Finally, in several problems of physical interest, such as the gelation process, dilute random matrices play an important role.⁽¹⁵⁾ The methods and results discussed here will be relevant for these problems too.

This paper presents two independent results for the Langevin dynamics of spin systems. In the first part we derive several general relations between many-point correlations and linear responses that any system in equilibrium with a thermal bath must respect. These equations are derived using a general fluctuation principle. We then discuss how these relations might be modified in a system evolving slowly out of equilibrium as we know the above-mentioned models do when the temperature of the environment is sufficiently low. In the second part of the paper we introduce a functional method to attack the Langevin dynamics of spin models defined on random graphs. We discuss how one recovers the well-known Schwinger–Dyson dynamic equations for the disordered-averaged two-time correlations and linear responses in the fully-connected limit. We next explain a set of iterations that allow us to deal with the disordered-averaged dynamics of the dilute problems in an approximated way. This part extends results briefly presented in ref. 16.

We wish to stress here that since we expect self-averageness when an infinite system evolves out of equilibrium, a model with a typical realization of disorder should behave in the same way as the averaged result predicts. Still, we also know that the infinite-size model will keep local fluctuations due to the sample-dependent fluctuations in the site connectivities and the random exchanges that will be lost when performing the disorder average.

The paper is organized as follows. In Section 2 we discuss the properties of many-time correlations in and out of equilibrium. We then argue on which are the modifications of these relations (applied to global functions) that are expected in a system that slowly evolves out of equilibrium. In Section 3 we give a precise definition of the models we are interested in and we introduce the functional method. Finally, we present our conclusions and directions for future work.

2. MANY-POINT FUNCTIONS

One way to characterize the dynamics of a generic model is to determine the evolution of the many-point correlators and linear responses defined as

$$C(i_1, t_1, \dots, i_k, t_k) = \langle s_{i_1}(t_1) \cdots s_{i_k}(t_k) \rangle, \quad (1)$$

$$R(i_1, t_1, \dots, i_{k-1}, t_{k-1}; i_k, t_k) = \frac{\delta \langle s_{i_1}(t_1) \cdots s_{i_{k-1}}(t_{k-1}) \rangle}{\delta h_{i_k}(t_k)} \Big|_{\hbar=0}, \quad (2)$$

$$R(i_1, t_1, \dots, i_{k-2}, t_{k-2}; i_{k-1}, t_{k-1}, i_k, t_k) = \frac{\delta^2 \langle s_{i_1}(t_1) \cdots s_{i_{k-2}}(t_{k-2}) \rangle}{\delta h_{i_{k-1}}(t_{k-1}) \delta h_{i_k}(t_k)} \Big|_{\hbar=0}, \quad (3)$$

etc. The angular brackets denote an average over different realizations of the dynamics. For a Langevin process such as the one in Eq. (6) this simply indicates an average over thermal noise realizations. The response in Eq. (2) corresponds to the case in which the magnetic field $h_{i_k}(t_k)$ couples linearly and instantaneously to the i_k th spin modifying the Hamiltonian at time t_k in such a way that $H \rightarrow H - h_{i_k}(t_k) s_{i_k}$. Equation (3) defines the response of the observable made of a product of $k-2$ spins to kicks applied linearly to two spins i_{k-1} and i_k at the instants t_{k-1} and t_k , respectively. One can easily generalize this definition to any number of kicks. Note that we do not assume any special ordering of times and that the kicked and responsive spins can be the same.

When studying models with quenched disorder, such as the ones introduced in Section 3, one is usually interested in their disorder-averaged behavior. Hence one further averages these expressions over the probability distribution of disorder and indicates this calculation by enclosing the right-hand-sides in square brackets. Since under the disorder average cross-terms involving different spins typically vanish, one usually focuses on the “global” auto correlations and responses. In particular, the two-time ones are defined as

$$C(t_1, t_2) \equiv \frac{1}{N} \sum_{i=1}^N [\langle s_i(t_1) s_i(t_2) \rangle]_J, \quad (4)$$

$$R(t_1; t_2) \equiv \frac{1}{N} \sum_{i=1}^N \frac{\delta [\langle s_i(t_1) \rangle]_J}{\delta h_i(t_2)} \Big|_{\hbar=0}. \quad (5)$$

2.1. Properties in Equilibrium

In equilibrium the time-dependence of multi-point correlations and responses is constrained in several ways. These constraints are a consequence of the fact that the probability distribution of any configuration \mathcal{C} of a thermostated system in equilibrium is given by the time-independent Gibbs–Boltzmann distribution $P(\mathcal{C}) \propto \exp(-\beta H)$, with β the inverse temperature and H the Hamiltonian, and that causality is expected to hold.

(We set the Boltzmann constant k_B to one henceforth.) Moreover, model-independent relations between correlations and responses can be established and are the expression of the so-called fluctuation–dissipation theorem (FDT).

These equilibrium properties have been worked out in detail for two-time functions and they are easy to derive using a variety of methods. In particular, for a Langevin process with white-noise one can exploit its connection to a Fokker–Planck equation or one can use the super-symmetric (SUSY) version of the dynamic generating function using the symmetries of the action to constrain the properties of the observables (see, e.g., ref. 5 for a review). With both techniques one finds the following properties of two-time functions:

- (i) Time-translation invariance (TTI), $C(i_1, t_1, i_2, t_2) = \tilde{C}(i_1, i_2; t_1 - t_2)$, for any two sites and any two-times.
- (ii) Causality, $R(i_1, t_1; i_2, t_2) = 0$ if $t_2 > t_1$.
- (iii) Fluctuation–dissipation theorem, $\beta R(i_1, t_1; i_2, t_2) = \partial_{t_2} C(i_1, t_1, i_2, t_2)$, for $t_1 \geq t_2$.
- (iv) Reciprocity or Onsager relations, $\langle A(t) B(t') \rangle = \langle A(t') B(t) \rangle$, for any pair of observables A and B which are functions of the spins.

For fully-connected models in the thermodynamic limit one can easily prove that all many-point correlations decouple into products of two-time ones (if there is an applied magnetic field, a one-time quantity, the average spin, also enters into the decomposition). Hence, these quantities have been the focus of most of not only the analytic but also the numeric and experimental studies of glassy systems.

Here, we establish generic properties that multi-point correlations and responses must satisfy in equilibrium. We shall later discuss how these may be generalized for systems slowly evolving out of equilibrium. In order to prove these generic properties we use the Fokker–Planck representation of the Langevin equation. As special cases we recover properties (i)–(iv) of two-time functions.

We focus on a stochastic dynamics of Langevin type in which each dynamic variable, s_j , evolves with

$$\partial_t s_j(t) = -\frac{\delta H}{\delta s_j(t)} + \xi_j(t). \quad (6)$$

$\xi_j(t)$ is a Gaussian thermal noise with zero mean and white-noise statistics:

$$\langle \xi_j(t) \xi_k(t') \rangle = 2T \delta_{jk} \delta(t - t'), \quad (7)$$

with T the temperature of the thermal bath in contact with the system. We have rescaled time in such a way that the friction coefficient is set to one. The time-dependent distribution function $P(\mathcal{C}, t) = P(\vec{s}, t)$ evolves according to the Fokker–Planck equation:

$$\frac{\partial}{\partial t} P(\vec{s}, t) = \sum_j \frac{\partial}{\partial s_j} \left[\frac{\delta H}{\delta s_j(t)} P(\vec{s}, t) + T \frac{\partial}{\partial s_j} P(\vec{s}, t) \right]. \quad (8)$$

In equilibrium this equation is solved by the Gibbs–Boltzmann distribution $P_{\text{eq}}(\vec{s}) = Z^{-1} e^{-\beta H(\vec{s})}$, with Z the partition function. A very useful representation of this equation takes advantage of its similarity with the Schrödinger equation and associates the functions of the stochastic variables with states in quantum mechanics⁽¹⁷⁾

$$f(\vec{s}) \leftrightarrow |f\rangle, \quad \langle g | f \rangle = \int d\vec{s} g(\vec{s})^* f(\vec{s}), \quad (9)$$

and defines position and momentum operators

$$s_j f(\vec{s}) \leftrightarrow \hat{s}_j |f\rangle, \quad -i \frac{\partial}{\partial s_j} f(\vec{s}) \leftrightarrow \hat{p}_j |f\rangle, \quad [\hat{s}_j, \hat{p}_k] = i\delta_{jk}. \quad (10)$$

Note that $\hat{s}_j^+ = \hat{s}_j$ and $\hat{p}_j^+ = \hat{p}_j$. Using this notation the Fokker–Planck equation reads

$$\frac{\partial}{\partial t} |P(t)\rangle = H_{\text{FP}} |P(t)\rangle, \quad |P(t)\rangle = e^{H_{\text{FP}} t} |P(0)\rangle, \quad H_{\text{FP}} = \sum_j \hat{p}_j (iH_j(\vec{s}) - T\hat{p}_j),$$

where $H_j \equiv \delta H / \delta s_j$. One also defines a projection state:

$$\langle - | f \rangle = \int d\vec{s} f(\vec{s}) \quad (11)$$

and writes the equilibrium state as $|P_{\text{eq}}\rangle = Z^{-1} e^{-\beta H(\vec{s})} |-\rangle$. The matrix elements of $\exp(H_{\text{FP}} t)$ are transition probabilities. Hence, the correlation functions of the observables A_i that are functions of the variables s_i are expressed as

$$\langle A_n(t_n) \cdots A_1(t_1) \rangle = \langle - | \hat{A}_n e^{H_{\text{FP}}(t_n - t_{n-1})} \hat{A}_{n-1} \cdots \hat{A}_1 e^{H_{\text{FP}} t_1} |P(0)\rangle, \quad (12)$$

with $t_n > t_{n-1} > \cdots > t_1 > 0$. \hat{A}_i is obtained from A_i by replacing the variables with the corresponding operators. The linear response to a field coupled linearly to the spin corresponds to adding $\sum_j -ih_j \hat{p}_j$ to H_{FP} . Thus

$\delta/\delta h_j(t) \leftrightarrow -i\hat{p}_j$ in the sense that the effect of the applied field is represented by the insertion of $-i\hat{p}_j$ at time t in a correlator.

Two useful properties also expected are time-reversal symmetry

$$e^{\beta H(\hat{s})} H_{\text{FP}} e^{-\beta H(\hat{s})} = H_{\text{FP}}^+, \quad (13)$$

which is due to detailed balance, and causality, which implies $\langle - | \hat{p}_i = 0$, so that the response of any correlator to a field applied after the observation times vanishes.

Let us now describe in detail the extensions of properties (i)–(iv) to multi-time functions.

2.1.1. Time Translation Invariance

The time-translation invariance of equilibrium correlation functions is transparent in this formalism. Consider Eq. (12) where all times are shifted by the same amount Δt :

$$\langle A_n(t_n + \Delta t) \cdots A_1(t_1 + \Delta t) \rangle = \langle - | \hat{A}_n e^{H_{\text{FP}}(t_n - t_{n-1})} \hat{A}_{n-1} \cdots \hat{A}_1 e^{H_{\text{FP}}(t_1 + \Delta t)} | P(0) \rangle. \quad (14)$$

If the system is equilibrated at time $t = 0$, $|P(0)\rangle = |P_{\text{eq}}\rangle$ and by definition $H_{\text{FP}} |P_{\text{eq}}\rangle = 0$ so that $\exp(H_{\text{FP}} \Delta t) |P_{\text{eq}}\rangle = |P_{\text{eq}}\rangle$. Hence the time-translation invariance of equilibrium correlation functions:

$$\langle A_n(t_n + \Delta t) \cdots A_1(t_1 + \Delta t) \rangle = \langle A_n(t_n) \cdots A_1(t_1) \rangle. \quad (15)$$

Clearly, for a two-time function one recovers property (i).

2.1.2. Generalized Onsager Relations

These express the time-reversal symmetry of equilibrium correlation functions. Consider $t_n > t_{n-1} > \cdots > t_1 = 0$ and, for simplicity, let us use the spins themselves as the observables. Then,

$$\langle s_{j_n}(t_n) \cdots s_{j_1}(0) \rangle = \langle - | \hat{s}_{j_n} e^{H_{\text{FP}}(t_n - t_{n-1})} \hat{s}_{j_{n-1}} \cdots \hat{s}_{j_2} e^{H_{\text{FP}} t_2} \hat{s}_{j_1} | P_{\text{eq}} \rangle \quad (16)$$

$$= \langle P_{\text{eq}} | \hat{s}_{j_1} e^{H_{\text{FP}}^+ t_2} \hat{s}_{j_2} \cdots \hat{s}_{j_{n-1}} e^{H_{\text{FP}}^+(t_n - t_{n-1})} \hat{s}_{j_n} | - \rangle. \quad (17)$$

Using $\langle P_{\text{eq}} | = \langle - | \exp(-\beta H)$, $| - \rangle = \exp(\beta H) | P_{\text{eq}} \rangle$ and relation (13), one obtains

$$\begin{aligned} & \langle s_{j_n}(t_n) s_{j_{n-1}}(t_{n-1}) \cdots s_{j_2}(t_2) s_{j_1}(0) \rangle \\ & = \langle s_{j_1}(t_n) s_{j_2}(t_n - t_2) \cdots s_{j_{n-1}}(t_n - t_{n-1}) s_{j_n}(0) \rangle. \end{aligned} \quad (18)$$

The usual Onsager relation on two-point functions is a particular case and reads: $\langle s_j(t) s_k(0) \rangle = \langle s_k(t) s_j(0) \rangle$. For a three-time correlator one has $\langle s_l(-t_2) s_k(t_1) s_j(t_2) \rangle = \langle s_j(-t_2) s_k(-t_1) s_l(t_2) \rangle$, for $t_2 > |t_1|$. If $l = j$, with the two extreme times fixed, the correlation is an even function of t_1 .

Similar relations can be derived for generic functions of the spins as long as they do not include, when expressed in the quantum mechanical language, the operator \hat{p} . In the two-time case these reduce to property (iv).

2.1.3. FD Relation on Two-Time Functions

According to the previous definitions, the usual equilibrium linear response function reads:

$$R_{jk}(t) = \frac{\delta}{\delta h_k(0)} \langle s_j(t) \rangle \Big|_{h=0} = \langle - | \hat{s}_j e^{H_{\text{FP}} t} (-i \hat{p}_k) | P_{\text{eq}} \rangle, \quad (19)$$

where we again simplify the presentation by considering the response of the simple observable given by a spin to a perturbation done on another spin. Generalizations to more complicated observables are straightforward.

On the other hand, the equilibrium correlation function and its time derivative are

$$C_{jk}(t) = \langle s_j(t) s_k(0) \rangle = \langle - | \hat{s}_j e^{H_{\text{FP}} t} \hat{s}_k | P_{\text{eq}} \rangle, \quad (20)$$

$$\frac{d}{dt} C_{jk}(t) = \langle - | \hat{s}_j e^{H_{\text{FP}} t} H_{\text{FP}} \hat{s}_k | P_{\text{eq}} \rangle. \quad (21)$$

By definition of the equilibrium probability distribution, $(Ti\hat{p}_k + H_k(\hat{S})) | P_{\text{eq}} \rangle = 0 \forall k$. Using this relation and the commutation properties of the operators, one obtains

$$H_{\text{FP}} \hat{s}_k | P_{\text{eq}} \rangle = Ti\hat{p}_k | P_{\text{eq}} \rangle \quad (22)$$

and thus recovers the usual FD relation in equilibrium:

$$R_{jk}(t) = -\frac{1}{T} \frac{d}{dt} C_{jk}(t), \quad t > 0. \quad (23)$$

2.1.4. FD Relation on 3-Point Functions

Response to a Single Kick

The first type of three-point response function one has to investigate is the response of a two-point correlator to a single kick. It is clear by causality that the response vanishes if the kick is posterior to the observation times. Two cases must then be distinguished:

• The perturbation is done prior to the two observation times. Let us choose $t_2 > t_1 > t_0$. Then

$$\frac{\delta}{\delta h_j(t_0)} \langle s_k(t_1) s_l(t_2) \rangle = \langle - | \hat{s}_l e^{H_{\text{FP}}(t_2-t_1)} \hat{s}_k e^{H_{\text{FP}}(t_1-t_0)} (-i\hat{p}_j) | P_{\text{eq}} \rangle. \quad (24)$$

The three-time correlator and its derivative with respect to the earlier time are:

$$\langle s_j(t_0) s_k(t_1) s_l(t_2) \rangle = \langle - | \hat{s}_l e^{H_{\text{FP}}(t_2-t_1)} \hat{s}_k e^{H_{\text{FP}}(t_1-t_0)} \hat{s}_j | P_{\text{eq}} \rangle, \quad (25)$$

$$\frac{\partial}{\partial t_0} \langle s_j(t_0) s_k(t_1) s_l(t_2) \rangle = - \langle - | \hat{s}_l e^{H_{\text{FP}}(t_2-t_1)} \hat{s}_k e^{H_{\text{FP}}(t_1-t_0)} H_{\text{FP}} \hat{s}_j | P_{\text{eq}} \rangle. \quad (26)$$

Using (22), one obtains

$$\frac{1}{T} \frac{\partial}{\partial t_0} \langle s_j(t_0) s_k(t_1) s_l(t_2) \rangle = \frac{\delta}{\delta h_j(t_0)} \langle s_k(t_1) s_l(t_2) \rangle \quad (27)$$

for local quantities. This implies the following identity for global ones:

$$\frac{1}{T} \partial_{t_0} C(t_2, t_1, t_0) = R(t_2, t_1; t_0). \quad (28)$$

This relation is a natural generalization of the usual FDT.

• The perturbation time is in between the two observation times. For the sake of clarity let us denote the three times $t_2 > t_1 > -t_2$. By time-translation invariance, we do not lose any generality with this choice. Then,

$$\frac{\delta}{\delta h_k(t_1)} \langle s_j(-t_2) s_l(t_2) \rangle = \langle - | \hat{s}_l e^{H_{\text{FP}}(t_2-t_1)} (-i\hat{p}_k) e^{H_{\text{FP}}(t_1+t_2)} \hat{s}_j | P_{\text{eq}} \rangle. \quad (29)$$

Considering the time-reversed (conjugated) expression, with j and l exchanged and t_1 reversed, and using the Baker–Campbell–Hausdorff formula $e^{-\beta H} \hat{p}_k e^{\beta H} = \hat{p}_k - i\beta H_k$, this can be rewritten as

$$\frac{\delta}{\delta h_k(-t_1)} \langle s_l(-t_2) s_j(t_2) \rangle = \langle - | \hat{s}_l e^{H_{\text{FP}}(t_2-t_1)} (i\hat{p}_k + \beta H_k) e^{H_{\text{FP}}(t_2+t_1)} \hat{s}_j | P_{\text{eq}} \rangle. \quad (30)$$

Consider now the derivative with respect to the intermediate time of the three-time correlator:

$$\frac{\partial}{\partial t_1} \langle s_j(-t_2) s_k(t_1) s_l(t_2) \rangle = \langle - | \hat{s}_l e^{H_{FP}(t_2-t_1)} [\hat{s}_k, H_{FP}] e^{H_{FP}(t_1+t_2)} \hat{s}_j | P_{eq} \rangle. \quad (31)$$

Since $[\hat{s}_k, H_{FP}] = -2iT\hat{p}_k - H_k$, we have the following *model-independent* FD relation:

$$\begin{aligned} \frac{1}{T} \frac{\partial}{\partial t_1} \langle s_j(-t_2) s_k(t_1) s_l(t_2) \rangle &= \frac{\delta}{\delta h_k(t_1)} \langle s_j(-t_2) s_l(t_2) \rangle \\ &\quad - \frac{\delta}{\delta h_k(-t_1)} \langle s_l(-t_2) s_j(t_2) \rangle. \end{aligned} \quad (32)$$

Note that while on the left-hand side we have the ‘‘expected’’ variation of the three-time correlator, on the right-hand side two responses appear (and none vanishes due to causality). As before, we can easily derive the consequences of Eq. (32) on global quantities:

$$\frac{1}{T} \partial_{t_1} C(t_2, t_1, -t_2) = R(t_2, -t_2; t_1) - R(t_2, -t_2; -t_1). \quad (33)$$

Response to Two Kicks

One can also construct the response of an observable to two earlier kicks. Taking $t_2 > t_1 > t_0$,

$$\frac{\delta^2 \langle s_l(t_2) \rangle}{\delta h_j(t_0) \delta h_k(t_1)} = \langle - | \hat{s}_l e^{H_{FP}(t_2-t_1)} (-i\hat{p}_k) e^{H_{FP}(t_1-t_0)} (-i\hat{p}_j) | P_{eq} \rangle, \quad (34)$$

and using Eq. (22) one obtains

$$\begin{aligned} \frac{\delta^2 \langle s_l(t_2) \rangle}{\delta h_j(t_0) \delta h_k(t_1)} &= \frac{1}{T} \frac{\partial}{\partial t_0} \left(\frac{\delta}{\delta h_k(t_1)} \langle s_j(t_0) s_l(t_2) \rangle \right) \\ &= \frac{1}{T^2} \frac{\partial^2}{\partial t_0 \partial t_1} \langle s_j(t_0) s_k(t_1) s_l(t_2) \rangle \\ &\quad + \frac{1}{T} \frac{\partial}{\partial t_0} \left(\frac{\delta}{\delta h_k(t_0+t_2-t_1)} \langle s_l(t_0) s_j(t_2) \rangle \right). \end{aligned} \quad (35)$$

Interestingly enough, we find that the three-point correlator does not determine completely the three-point responses.

2.2. A General Fluctuation Principle

In the previous section we derived some FD relations on a case-by-case basis. One may wonder whether these relations can be expressed in a unified way.

For the sake of simplicity, let us consider the case of a single time-varying field $h(t)$, coupled linearly to any given observable $\mathcal{O}(s)$ (here we denote by s the whole spin configuration: $s \equiv \{s_1 \cdots s_N\}$). In order to state a unifying principle for the multi-time FD relations, we need some quantity which encodes the full hierarchy of correlation and response functions defined above. Let us denote by $\underline{x}_{-t_M, t_M}$ a particular trajectory of the observable x between the times $-t_M$ and t_M . Hereafter we shall assume t_M to be larger than any other time in the problem and we shall drop the subscripts. Consider the probability density $dP(\underline{s} | \underline{h})$ of a trajectory \underline{s} of the system, given a particular realization \underline{h} of the external field. It is clear that, by integrating $dP(\underline{s} | \underline{h})$ over \underline{s} and expanding in powers of \underline{h} , we can recover all the correlation and response functions.

In order to state a fluctuation principle for $P(\underline{s} | \underline{h})$, we need to define the time-reversal operation. The reversed time of \underline{x} is denoted by \underline{x}^R and is defined by $x^R(t) \equiv x(-t)$. We make the following assumptions: (i) the probability distribution at the initial time $-t_M$ is the equilibrium one in zero-field: $P_{\text{eq}}(s)$; (ii) the dynamics satisfies detailed balance; (iii) the perturbing field vanishes outside the time interval $[-t_M, t_M]$. Under these hypotheses, it is easy to show that

$$\frac{dP(\underline{s}^R | \underline{h}^R)}{dP(\underline{s} | \underline{h})} = \exp \left\{ -\beta \int_{-t_M}^{t_M} dt h(t) \dot{\mathcal{O}}(t) \right\}, \quad (36)$$

where $\dot{\mathcal{O}}(t)$ denotes the time derivative of the observable $\mathcal{O}(s)$ along the trajectory \underline{s} .

The proof of Eq. (36) is straightforward³ if we assume Ito discretization of the Langevin dynamics (6). In this case the probability density of a trajectory \underline{s} can be written explicitly:

$$dP(\underline{s} | \underline{h}) = P_{\text{eq}}(s(-t_M)) \left\langle \prod_{i,t} \delta \left(\xi_i(t) - \dot{s}_i(t) - \frac{\delta H}{\delta s_i} + h(t) \frac{\delta \mathcal{O}}{\delta s_i} \right) \right\rangle_{\xi} \cdot d\underline{s}, \quad (37)$$

³ Nevertheless the result (36) holds under more general hypotheses, for instance in a discrete-time Markov chain.

where $P_{\text{eq}}(s)$ is the equilibrium distribution. Using this expression one obtains

$$\frac{dP(\underline{s}^R | \underline{h}^R)}{dP(\underline{s} | \underline{h})} = \frac{P_{\text{eq}}(s(t_M))}{P_{\text{eq}}(s(-t_M))} \exp \left\{ -\beta \sum_i \int_{-t_M}^{t_M} dt \left[-\frac{\delta H}{\delta s_i} + h(t) \frac{\delta \mathcal{O}}{\delta s_i} \right] \dot{s}_i(t) \right\}, \quad (38)$$

which reduces to (36) upon insertion of the Boltzmann distribution $P_{\text{eq}}(s) \propto \exp(-\beta H(s))$.

It can be useful to formulate a few simple remarks on this result:

- In the case $\underline{h} = 0$, Eq. (36) is simply a rephrasing of time-reversal invariance. A time-varying external field violates this invariance. The amount of such a violation is quantified by the work done by the field on the system.

- Equation (36) can be regarded as the dynamic analogous to the following identity:

$$\frac{dP_{\text{eq}}(s | -h)}{dP_{\text{eq}}(s | h)} = \exp\{-2\beta h\mathcal{O}(s)\}, \quad (39)$$

which holds if $dP_{\text{eq}}(s | -h)$ is the probability of the configuration s according to the Boltzmann distribution.

- Using Eq. (36) we can recover the FD relations of the previous section. As a simple exercise, let us consider the relation between two-point functions (23). We take $\mathcal{O}(s) = s_k$, multiply both sides of Eq. (36) by $s_j(t_1) dP(\underline{s} | \underline{h})$ and integrate over \underline{s} . This yields

$$\langle s_j(-t_1) \rangle_{\underline{h}^R} = \left\langle s_j(t_1) \cdot \exp \left\{ -\beta \int dt' h_k(t') \dot{s}_k(t') \right\} \right\rangle_{\underline{h}}, \quad (40)$$

where $\langle \cdot \rangle_{\underline{h}}$ denotes the average under an applied magnetic field \underline{h} . Expanding the two members of this identity in powers of \underline{h} , we get, to linear order,

$$\frac{1}{T} \frac{\partial}{\partial t_2} \langle s_j(t_1) s_k(t_2) \rangle = \frac{\delta \langle s_j(t_1) \rangle}{\delta h_k(t_2)} - \frac{\delta \langle s_j(-t_1) \rangle}{\delta h_k(-t_2)}, \quad (41)$$

which, using causality, yields back Eq. (23).

The general strategy for deriving FD relations for multi-time functions is easily stated: (i) multiply both sides of Eq. (36) by the quantity $s_{i_1}(t_1) \cdots s_{i_m}(t_m) \cdot dP(\underline{s} | \underline{h})$; (ii) integrate over \underline{s} ; (iii) expand in powers of \underline{h}

and collect the terms multiplying $h(t_{m+1}) \cdots h(t_{m+n})$. This procedure yields a relation for $(m+n)$ -times functions. Of course if we want to study the response to kicks on several distinct spins we must consider the obvious generalization of Eq. (36) to the case of several observables.

- Equation (36) can be used to derive relations which are exact with a non-vanishing perturbing field. A simple example is Eq. (40). Unlike for the usual FD theorem (23), one is not obliged to take the zero-field limit, which can be numerically tricky.

- Finally, the principle (36) is quite reminiscent of the Gallavotti–Cohen (GC) theorem^(18, 19) as stated in ref. 20 (see also refs. 21 and 22) for stochastic dynamics. However GC refers to stationary systems and has a non-trivial content only if the dynamics violates detailed balance. In Eq. (36) we consider the complementary situation. Detailed balance is satisfied at any time, but time-reversal invariance is violated by the explicit time-dependence of the external field. Hopefully this type of result is more suitable for the present context.

2.3. Extensions Out of Equilibrium

The non-equilibrium slow dynamics of glassy systems presents rather simple modifications of the equilibrium properties of two-time correlations and responses.^(5, 7, 23) The two-time global correlations decay (increase) monotonically as a function of the longer (shorter) time. Therefore, one can invert the two-time correlation relation to write

$$C(t_1, t_3) = f(C(t_1, t_2), C(t_2, t_3); t_3), \quad (42)$$

for any triplet of times t_1, t_2, t_3 . The function $f(C_{12}, C_{23}; t_3)$ is then assumed to have a limit as $t_3 \rightarrow \infty$ with the first two arguments fixed and finite. We can drop the last argument and write “triangular relations” that link, in the limit of long times, any two-time correlation to two others evaluated at an intermediate time:⁽²³⁾

$$C(t_1, t_3) \approx f(C(t_1, t_2), C(t_2, t_3)), \quad t_1 \geq t_2 \geq t_3. \quad (43)$$

From Eq. (43) one can derive strong constraints on the form of the correlation function.⁽²³⁾ In the simplest scenario one gets two types of behaviors, depending on how the long-time limit is taken. In the “quasi-equilibrium” regime the time interval $t_2 - t_1 \equiv \tau$ is kept fixed:

$$\lim_{t_1, t_2 \rightarrow \infty} C(t_1, t_2) = F(\tau). \quad (44)$$

As τ grows from 0 to ∞ , $F(\tau)$ decreases from 1 to the Edwards–Anderson parameter $q_{\text{EA}} > 0$.

The “aging” regime is characterized in terms of a monotonously increasing function $l(t)$, such that $\lim_{t \rightarrow \infty} l(t) = \infty$. One then takes the long-time limit keeping the ratio $l(t_2)/l(t_1) \equiv z$ fixed:

$$\lim_{t_1, t_2 \rightarrow \infty} C(t_1, t_2) = \mathcal{F}(z). \quad (45)$$

The function $\mathcal{F}(z)$ decreases from q_{EA} to 0 as its argument increases from 1 to ∞ .

Finally, in the aging limit, one finds that the FDT is modified to

$$R(t_1; t_2) = \theta(t_1 - t_2) \frac{1}{T_{\text{EFF}}(C(t_1, t_2))} \frac{\partial C(t_1, t_2)}{\partial t_2}, \quad (46)$$

with T_{EFF} a correlation-scale-dependent effective temperature.⁽²⁴⁾ In this section we discuss possible generalizations of these properties to the case of multi-time functions.

2.3.1. Time-Scalings

Let us first discuss the consequences of having multiple correlation scales, as defined via the behavior of the two-time global correlator, on the multi-time *global* ones. The latter are defined from Eqs. (4):

$$C(t_1, t_2, \dots, t_n) \equiv \frac{1}{N} \sum_i C(i, t_1, i, t_2, \dots, i, t_n), \quad (47)$$

and are the ones expected to be relevant in a disorder-averaged treatment.

For concreteness, take a system with two correlation scales (as happens for the fully-connected p spin model and, presumably, for the dilute case too). This means that in the asymptotic limit in which $t_1 \geq t_2$ are both long but their ratio $l(t_1)/l(t_2)$ varies between one and infinity the times are classified according to the value of $C(t_1, t_2)$. If t_1 and t_2 are nearby and $C(t_1, t_2) \geq q_{\text{EA}}$, we are in the fast correlation scale. If $l(t_1)$ and $l(t_2)$ are far away, $C(t_1, t_2) < q_{\text{EA}}$, and we are in the slow correlation scale.

Taking three times $t_1 \geq t_2 \geq t_3$ we have four possibilities:

(i) The three times are nearby with all correlations being larger than q_{EA} .

(ii) The longer times t_1 and t_2 are nearby while the shortest one t_3 is far away; in this case, $C(t_1, t_2) \geq q_{\text{EA}} \geq C(t_1, t_3)$ and $C(t_1, t_2) \geq q_{\text{EA}} \geq C(t_2, t_3)$.

(iii) The “reversed” situation in which the two shorter times t_2 and t_3 are nearby and the longest one t_1 is far away; then $C(t_1, t_2) \leq q_{\text{EA}} \leq C(t_2, t_3)$ and $C(t_1, t_3) \leq q_{\text{EA}} \leq C(t_2, t_3)$.

(iv) Finally, the three times can be far away from each other, in which case all correlations are smaller than q_{EA} .

The property of monotonicity can be used to express any multi-time correlator in terms of two-time ones. Take, for instance, a correlator evaluated at three times, $C(t_1, t_2, t_3)$. Using the monotonicity property one can invert the two-time correlation $C_{12} \equiv C(t_1, t_2)$ and write $t_1 = g(C_{12}, t_2)$. Equivalently, $t_2 = g(C_{23}, t_3)$. Thus,

$$C(t_1, t_2, t_3) = C(g(C_{12}, g(C_{23}, t_3)), g(C_{23}, t_3), t_3). \quad (48)$$

If we assume that the limit $t_1 \geq t_2 \geq t_3 \rightarrow \infty$ while C_{12} and C_{23} are kept fixed, then there exists

$$C(t_1, t_2, t_3) = f_1^{(3)}(C_{12}, C_{23}). \quad (49)$$

Clearly, we could have chosen to work with $t_1 = g(C_{13}, t_3)$ and obtain

$$C(t_1, t_2, t_3) = f_2^{(3)}(C_{13}, C_{23}). \quad (50)$$

The liberty to choose a representation is clear in the equilibrium case where one can write

$$C(t_1, t_2, t_3) = f_{\text{eq},1}^{(3)}(t_1 - t_2, t_2 - t_3) = f_{\text{eq},2}^{(3)}(t_1 - t_3, t_2 - t_3) = f_{\text{eq},3}^{(3)}(t_1 - t_3, t_1 - t_2).$$

With similar arguments one can write any multi-time correlation in terms of two-time correlations only admitting that the limits exist. In all cases we choose to take the limit of the shortest time to infinity to ensure that all two-time correlations are in their asymptotic regime.

As an example, consider the case in which there are two correlation scales and the three times are chosen in such a way that two of them, t_1 and t_2 , are nearby so that the two-time correlation $C(t_1, t_2)$ falls above q_{EA} and the third one is far away from both in such a way that the two-time correlations $C(t_1, t_3)$ and $C(t_2, t_3)$ fall below q_{EA} . Then we expect

$$C(t_1, t_2, t_3) = \tilde{f} \left(t_1 - t_2, \frac{l(t_2)}{l(t_3)} \right).$$

This relation (and similar ones) generalize time-translation invariance for multi-time correlations [see Eq. (15)] to the slowly evolving non-equilibrium case.

2.3.2. Fluctuation–Dissipation Relations

Another important feature of glassy dynamics is the modification of the fluctuation–dissipation theorem. The corresponding out-of-equilibrium fluctuation–dissipation relation (OFDR) for two-point functions has been the object of intensive studies in recent years (see ref. 5 for a review). In Section 2.1.4 we obtained fluctuation–dissipation relations for multi-time correlations and responses. Here, we discuss the possible form of the corresponding multi-time OFDR's.

Guessing the correct generalization of the multi-time FD relations is quite difficult. Already in equilibrium, the form of these relations is far from obvious and a careful derivation is necessary. Here we shall adopt the following approach. We consider the multi-point correlation and response functions of a Gaussian model. In this case the two-time OFDR straightforwardly implies multi-time OFDR's. We then rewrite these relations in a model-independent way. This can be done in a particular compact way by modifying the fluctuation principle of Section 2.2.

For the sake of simplicity, we shall consider the case of a scalar field $\phi(t)$ with $t \geq 0$, linearly coupled to an external field $h(t)$. Being Gaussian, its behavior is completely specified by the following quantities:

$$M(t) = \langle \phi(t) \rangle, \quad C(t, t') = \langle \phi(t) \phi(t') \rangle, \quad R(t; t') = \left. \frac{\delta \langle \phi(t) \rangle}{\delta h(t')} \right|_{h=0}. \quad (51)$$

As a warm-up exercise, let us consider three-time functions. It is simple to show that, in the Gaussian case:

$$C(t_1, t_2, t_3) = M(t_1) M(t_2) M(t_3) + M(t_1) C(t_2, t_3) + M(t_2) C(t_1, t_3) + M(t_3) C(t_1, t_2), \quad (52)$$

$$R(t_2, t_3; t_1) = M(t_3) R(t_2; t_1) + M(t_2) R(t_3; t_1), \quad (53)$$

$$R(t_3; t_2, t_1) = 0, \quad (54)$$

where we chose the ordering of times $t_1 < t_2 < t_3$, which we shall keep in this section. We should now make some assumptions on the long-time behavior of the functions in Eq. (51). To keep the presentation as simple as possible we shall consider a scenario with two correlation scales, cf. Eqs. (44) and (45):

$$M(t) \approx M_{\text{eq}}, \quad (55)$$

$$C(t, t') \approx C_{\text{eq}}(t-t') + C_{\text{ag}}(l(t)/l(t')), \quad (56)$$

$$R(t; t') \approx R_{\text{eq}}(t-t') + \tilde{l}(t') R_{\text{ag}}(l(t)/l(t')), \quad (57)$$

with

$$\beta \partial_\tau C_{\text{eq}}(\tau) = -R_{\text{eq}}(\tau), \quad \beta_{\text{EFF}} \partial_{t'} C_{\text{ag}}(l(t)/l(t')) = \tilde{l}(t') R_{\text{ag}}(l(t)/l(t')). \quad (58)$$

We defined $\tilde{l}(t) = l'(t)/l(t)$, which we assume to vanish when $t \rightarrow \infty$. All the times in our discussion below are such that the above asymptotic forms are well verified.

Let us consider the different cases for the response to a single kick (the two-kick case can be analysed along the same lines):

- The perturbation is done at time t_1 , i.e., prior to the two observation times. We must distinguish several different possibilities according to the scaling of the time separations $(t_2 - t_1)$ and $(t_3 - t_2)$, as $t_1, t_2, t_3 \rightarrow \infty$:

– $(t_2 - t_1)$ and $(t_3 - t_2)$ of $O(1)$. We get

$$\partial_{t_1} C(t_1, t_2, t_3) \approx -M_{\text{eq}} \partial C_{\text{eq}}(t_3 - t_1) - M_{\text{eq}} \partial C_{\text{eq}}(t_2 - t_1), \quad (59)$$

$$R(t_2, t_3; t_1) \approx M_{\text{eq}} R_{\text{eq}}(t_3 - t_1) + M_{\text{eq}} R_{\text{eq}}(t_2 - t_1), \quad (60)$$

where we denoted $\partial C(\cdot)$ as the derivative of $C(\cdot)$ with respect to its argument. From Eqs. (59) and (60) we recover the FD relation for three-time functions:

$$\beta \partial_{t_1} C(t_1, t_2, t_3) \approx R(t_2, t_3; t_1), \quad (61)$$

which coincides with Eq. (28).

– $(t_2 - t_1) = O(1)$, $(t_3 - t_2) \rightarrow \infty$. In this case we have

$$\partial_{t_1} C(t_1, t_2, t_3) \approx -M_{\text{eq}} \frac{l(t_3) l'(t_1)}{l(t_1)^2} \partial C_{\text{ag}}(l(t_3)/l(t_1)) - M_{\text{eq}} \partial C_{\text{eq}}(t_2 - t_1), \quad (62)$$

$$R(t_2, t_3; t_1) \approx M_{\text{eq}} \tilde{l}(t_1) R_{\text{ag}}(l(t_3)/l(t_1)) + M_{\text{eq}} R_{\text{eq}}(t_2 - t_1). \quad (63)$$

At first glance it may seem that Eqs. (62) and (63) are much more difficult to deal with than Eqs. (59) and (60). Notice, however, that the first terms in the above expressions are of order $\tilde{l}(t_1)$ with respect to the second ones, and can therefore be dropped in the aging limit. In this limit the FD relation (61) is once again satisfied.

– $(t_2 - t_1) \rightarrow \infty$, $(t_3 - t_2) = O(1)$. We have

$$\partial_{t_1} C(t_1, t_2, t_3) \approx M_{\text{eq}} \partial_{t_1} C_{\text{ag}}(l(t_3)/l(t_1)) + M_{\text{eq}} \partial_{t_1} C_{\text{ag}}(l(t_2)/l(t_1)), \quad (64)$$

$$R(t_2, t_3; t_1) \approx M_{\text{eq}} \tilde{l}(t_1) R_{\text{ag}}(l(t_3)/l(t_1)) + M_{\text{eq}} \tilde{l}(t_1) R_{\text{ag}}(l(t_2)/l(t_1)). \quad (65)$$

Using Eq. (58), it is easy to show that these functions satisfy the natural out-of-equilibrium generalization of Eq. (61):

$$\beta_{\text{EFF}} \partial_{t_1} C(t_1, t_2, t_3) \approx R(t_2, t_3; t_1). \quad (66)$$

– $(t_2 - t_1) \rightarrow \infty, (t_3 - t_2) \rightarrow \infty$. This case is similar to the previous one. The relation (66) is recovered.

• The perturbation time is t_2 , i.e., in between the two observation times. This case is more interesting than the previous one. As before, we need to treat separately the different time-scalings:

– $(t_2 - t_1), (t_3 - t_2) = O(1)$. This case is quite simple:

$$\partial_{t_2} C(t_1, t_2, t_3) \approx -M_{\text{eq}} \partial C_{\text{eq}}(t_3 - t_2) + M_{\text{eq}} \partial C_{\text{eq}}(t_2 - t_1), \quad (67)$$

$$R(t_1, t_3; t_2) \approx M_{\text{eq}} R_{\text{eq}}(t_3 - t_2), \quad (68)$$

$$R(t_1^R, t_3^R; t_2^R) \approx M_{\text{eq}} R_{\text{eq}}(t_2 - t_1), \quad (69)$$

where we defined the time-reversal operation as follows

$$t \mapsto t^R \equiv 2t^* - t \quad (70)$$

for some (large) fixed time t^* . Using Eq. (58) we re-obtain the equilibrium relation, see Eq. (33),

$$\beta \partial_{t_2} C(t_1, t_2, t_3) \approx R(t_1, t_3; t_2) - R(t_1^R, t_3^R; t_2^R). \quad (71)$$

– $(t_2 - t_1) = O(1), (t_3 - t_2) \rightarrow \infty$. We have

$$\partial_{t_2} C(t_1, t_2, t_3) \approx -M_{\text{eq}} \frac{l(t_3) l'(t_2)}{l(t_2)^2} \partial C_{\text{ag}}(l(t_3)/l(t_2)) + M_{\text{eq}} \partial C_{\text{eq}}(t_2 - t_1), \quad (72)$$

$$R(t_1, t_3; t_2) \approx M_{\text{eq}} \tilde{l}(t_2) R_{\text{ag}}(l(t_3)/l(t_2)), \quad (73)$$

$$R(t_1^R, t_3^R; t_2^R) \approx M_{\text{eq}} R_{\text{eq}}(t_2 - t_1). \quad (74)$$

Notice that the first contribution to the correlation, cf. Eq. (72), and the response (73) are of order $\tilde{l}(t_2)$ with respect to the other terms, and therefore vanish in the aging limit. One recovers, therefore, the equilibrium relation (71).

– $(t_2 - t_1) \rightarrow \infty, (t_3 - t_2) = O(1)$. This case works exactly as the previous one: the equilibrium FD relation (71) holds up to terms of relative order $\tilde{l}(t_1)$.

– $(t_2 - t_1), (t_3 - t_2) \rightarrow \infty$. Here something interesting finally happens:

$$\partial_{t_2} C(t_1, t_2, t_3) \approx M_{\text{eq}} \partial_{t_2} C_{\text{ag}}(l(t_3)/l(t_2)) + M_{\text{eq}} \partial_{t_2} C_{\text{ag}}(l(t_2)/l(t_1)), \quad (75)$$

$$R(t_1, t_3; t_2) \approx M_{\text{eq}} \tilde{l}(t_2) R_{\text{ag}}(l(t_3)/l(t_2)), \quad (76)$$

$$R(t_1^{RI}, t_3^{RI}; t_2^{RI}) \approx M_{\text{eq}} \tilde{l}(t_1) R_{\text{ag}}(l(t_2)/l(t_1)), \quad (77)$$

where we defined the ‘‘aging’’ time-reversal transformation as follows:

$$t \mapsto t^{RI} \equiv l^{-1} \left(\frac{l(t^*)^2}{l(t)} \right), \quad (78)$$

for some (large) fixed time t^* . Using this expression and the relations (58) one can easily show that

$$\beta_{\text{EFF}} \partial_{t_2} C(t_1, t_2, t_3) \approx R(t_1, t_3; t_2) - R(t_1^{RI}, t_3^{RI}; t_2^{RI}), \quad (79)$$

which should be compared with Eq. (33). In other words, the out-of-equilibrium FD relation is obtained from the equilibrium one by replacing the thermodynamic temperature with an effective one and by a *redefinition of the time-reversal operation*.

At this point it is easy to summarize the above results (and the similar ones which can be derived for higher correlations) along the lines of Section 2.2. More precisely, let us split the field into its fast components plus the slowly varying part:^(25, 26)

$$\phi(t) = \phi_{\text{eq}}(t) + \phi_{\text{ag}}(t), \quad h(t) = h_{\text{eq}}(t) + h_{\text{ag}}(t). \quad (80)$$

While $\phi_{\text{eq}}(t)$ obeys a fluctuation principle of the form (36), this has to be modified as follows for $\phi_{\text{ag}}(t)$:

$$\frac{dP(\underline{\phi}_{\text{ag}}^{RI} | \underline{h}_{\text{ag}}^{RI})}{dP(\underline{\phi}_{\text{ag}} | \underline{h}_{\text{ag}})} \approx \exp \left\{ -\beta_{\text{EFF}} \int dt h_{\text{ag}}(t) \dot{\phi}_{\text{ag}}(t) \right\}. \quad (81)$$

Here \underline{h}^{RI} and $\underline{\phi}^{RI}$ are defined analogously to \underline{h}^R and $\underline{\phi}^R$ [cf. Eq. (36)] but using the time-reversal operation (78) instead of the usual one. The reader can easily check Eq. (81) on a Gaussian process. It is natural to conjecture that it holds even for non-Gaussian ones. It would be of great interest to check its consequences in a numerical simulation.

Let us, by the way, note that Eq. (81) implies a whole class of out-of-equilibrium Onsager relations. A simple example is

$$C_{\text{ag}}(t_1, t_2, t_3) = C_{\text{ag}}(t_1, t'_2, t_3) \quad \text{if} \quad l(t_2)l(t'_2) = l(t_1)l(t_3). \quad (82)$$

In order to check numerically such a relation one could proceed as follows. For a given triplet of times $t_1 \ll t_2 \ll t_3$, the “reversed” time t'_2 is determined by $C(t_1, t_2) = C(t'_2, t_3)$. The three-time functions $C(t_1, t_2, t_3)$ and $C(t_1, t'_2, t_3)$ should then be compared.

3. DILUTE DISORDERED SPIN SYSTEMS

Dilute disordered spin models are typical cases where a description in terms of two-point functions only is not complete. Multi-point correlations do not factorize and carry non-trivial information about the dynamics of these systems.

In this section, we introduce a method to solve the disordered-averaged dynamics of these models with a succession of approximate steps. The approach follows closely the one used by Biroli and Monasson⁽²⁷⁾ for the analysis of the spectrum of random matrices. In short, we first define the models of interest, we next explain the functional method used and we finally give a hint on how the equations derived can be solved in some simple cases.

3.1. The Models

A family of disordered spin models is defined by the Hamiltonian

$$H_J = - \sum_{i_1 < \dots < i_p} J_{i_1 \dots i_p} s_{i_1} s_{i_2} \dots s_{i_p}. \quad (83)$$

The spins s_i , $i = 1, \dots, N$, can be Ising variables, $s_i = \pm 1$, $\forall i$ (Ising model) or they can be real variables. In the latter case one can either constrain them such that $\sum_{i=1}^N s_i^2 = N$ (spherical model) or add a “soft-spin” term to the Hamiltonian which favors the values of s_i around ± 1 . The parameter p is a fixed integer, $p \geq 2$, which controls the number of spins involved in each interaction term.

The couplings between the spins are given by the quenched random variables $J_{i_1 \dots i_p}$. In the fully-connected case, all the $\binom{N}{p}$ entries in these “tensors” are non-zero and, equivalently, the model is defined on the

complete (hyper) graph. Generally, one uses a Gaussian (or bimodal) probability distribution with zero mean and variance $[J_{i_1 \dots i_p}^2]_J = p! J_0^2 / (2N^{p-1})$ in order to ensure a good thermodynamic limit. (Henceforth, the square brackets denote an average over quenched disorder.)

In the dilute case, there is only an extensive (proportional to N) number of non-zero interaction terms in the sum (83). The interactions are drawn from

$$P(J_{i_1 \dots i_p}) = \left(1 - \frac{\alpha p!}{N^{p-1}}\right) \delta(J_{i_1 \dots i_p}) + \frac{\alpha p!}{N^{p-1}} \Pi(J_{i_1 \dots i_p}), \quad (84)$$

in which $\Pi(J)$ does not contain a Dirac distribution at $J_{i_1 \dots i_p} = 0$. The (finite) parameter α is the average ratio of the number of interacting p -uplets of spins per variable. When $p = 2$ this is the Viana–Bray model.^(30, 31) Its geometrical structure is that of the celebrated Erdős–Renyi random graph:⁽³²⁾ the model consists of spins occupying the vertices of a graph drawn from this ensemble, with interactions on the edges. For $p \geq 3$ the model is defined on a random hypergraph, the edges are replaced by plaquettes linking p vertices. For any p , the probability distribution of the degree of a given spin, i.e., the number of interactions it belongs to, is a Poisson law with mean αp .

As there is an extensive number of non-zero terms in (83) in the dilute case, each of them must be of order 1 to obtain a sensible thermodynamic limit. The distribution Π of the non-vanishing couplings must thus have finite mean and variance. For concreteness we choose

$$\Pi(J_{i_1 \dots i_p}) = \frac{1}{2} [\delta(J_{i_1 \dots i_p} - \tilde{J}) + \delta(J_{i_1 \dots i_p} + \tilde{J})], \quad (85)$$

with \tilde{J} finite. Note that the fully-connected limit is recovered from the dilute case by taking $\alpha = N^{p-1}/p!$ and $\tilde{J}^2 = p! J_0^2 / (2N^{p-1})$, which in the limit $N \rightarrow \infty$ become $\alpha \rightarrow \infty$, $\tilde{J} \rightarrow 0$ with $J_0^2 = 2\alpha \tilde{J}^2$ finite.

The soft-spin Ising (a) or spherical (b) constraint are imposed via an additional term in the Hamiltonian:

$$V_I = \sum_{j=1}^N \kappa (s_j^2 - 1)^2 \quad (a), \quad V_S = \mu(t) \sum_{j=1}^N (s_j^2 - 1) \quad (b), \quad (86)$$

The (time-independent) parameter κ should be taken to infinity to recover the Ising limit, and the (time-dependent) Lagrange multiplier $\mu(t)$ is to be determined self-consistently by requiring that the equal-time correlator be

normalised to one. In what follows we call H the full Hamiltonian, given by the sum of H_J and the appropriate constraint, $V = V_I$ or $V = V_S$,

$$H = H_J + V, \quad V \equiv \sum_j v(s_j). \quad (87)$$

Our aim is to study the Langevin dynamics of such a model, defined by Eqs. (6) and (7), which mimics the physical situation of a system in contact with a thermostat at fixed temperature.

3.2. Summary of Results

Several special cases of the model defined in Eqs. (6) and (83) have been studied in the past. The vast majority of these studies has been confined to the (technically simpler) case of fully-connected models. For a summary of results see ref. 5.

There has been, however, recent growing interest in dilute systems,^(30, 31, 33, 34) for at least two reasons. On the one hand, even if they are still mean-field, with no notion of geometry, their finite connectivity is an ingredient for the description of real physical systems that was absent from the fully-connected models. Moreover, the fluctuation of their local connectivity leads to Griffiths phases,⁽⁸⁾ which are experimentally observed in some physical systems. On the other hand, a large number of optimization problems, satisfiability for instance, can be mapped onto dilute models of Ising spins.^(11, 12) This observation triggered a large effort towards the understanding of the static properties of such models, which led to very interesting recent results.⁽¹³⁾ At the moment the understanding of their dynamic behavior is much poorer. One could hope, however, that it would be very useful for the study and improvement of local search algorithms that solve these optimization problems.⁽¹⁴⁾

Previous studies of physical dynamics of dilute p spin models include the following:

(i) Monte Carlo dynamic simulations were performed for Ising spins and two-body^(35, 36) or three-body⁽³⁷⁾ interactions. These numerical studies pointed out the heterogeneous character of the non-equilibrium dynamics for fixed quenched disorder. Moreover refs. 36 and 37 demonstrated the validity of out-of-equilibrium fluctuation–dissipation relations for single-spin two-time functions. The single-spin OFDR's turned out to agree with the results of a static calculation. (See ref. 38 for local FD relations between coarse-grained two-time quantities that are not disorder averaged in finite-dimensional glassy systems.)

(ii) An analytic solution to the disorder-averaged dynamics of the spherical $p = 2$ dilute model can be achieved by solving the Langevin equation in the rotated basis in which the interaction matrix is diagonal.⁽¹⁶⁾ This study showed the existence of two non-equilibrium dynamic asymptotic regimes. The first one is very similar to that of the fully-connected counterpart model.⁽³⁹⁾ The second one is dominated by the tails in the spectrum of the random interaction matrix and corresponds to progressive condensation on the eigenvectors that are localized on these sites.

Let us state as a side remark here that the spherical dilute model with $p \geq 3$ is pathological. Indeed, the spherical constraint $\sum_j s_j^2 = N$ can be satisfied in at least two opposite ways. Either all the spins are of order 1, which is the situation one would like to obtain, or a finite number of them are of order \sqrt{N} , a strongly localized situation. Imagine that the p spins interacting through a given plaquette are all localized. As the coupling on a plaquette is of order 1 in the dilute case, this would contribute a term of order $N^{p/2}$ to the energy; this situation thus dominates the thermodynamic limit of such models when $p \geq 3$. Note, however, that such a pathology can be cured by adding infinitesimal terms of the soft-spin type, $\epsilon \sum_j (s_j^2 - 1)^n$, with n sufficiently high, to the Hamiltonian.

3.3. Functional Formalism

In this section we introduce a generic formalism to derive macroscopic dynamic equations for the evolution of global correlation and response functions.

Standard techniques allow us to derive a dynamic generating functional, \mathcal{Z} , for a generic Langevin process with white noise, as a path integral:⁽⁴⁰⁾

$$\begin{aligned} \mathcal{Z}[\eta, \hat{\eta}] &= \int \mathcal{D}s_i \mathcal{D}i \hat{s}_i \mathcal{D}\bar{\psi}_i \mathcal{D}\psi_i \exp \left(-S_{\text{eff}} + \int_0^t dt' [\eta_i(t') s_i(t') - \hat{\eta}_i(t') i\hat{s}_i(t')] \right) \\ -S_{\text{eff}} &= \int_0^t dt' \left[T(i\hat{s}_i(t'))^2 + i\hat{s}_i(t') \left(\partial_{i s_i(t')} + \frac{\delta H}{\delta s_i(t')} \right) \right] \\ &\quad + \int_0^t dt' \int_0^t dt'' \bar{\psi}_i(t') \left(\delta(t' - t'') \delta_{ij} \partial_{t''} + \frac{\delta^2 H}{\delta s_i(t') s_j(t'')} \right) \psi_j(t''), \end{aligned} \quad (88)$$

where we have introduced two time-dependent sources, $\eta_i(t)$ and $\hat{\eta}_i(t)$. Einstein's summation convention is assumed. The fields s_i and \hat{s}_i are real, \hat{s}_i being a response field conjugated to s_i , whereas $\bar{\psi}_i$ and ψ_i are Grassman

fermionic fields. The correlation and response functions of the stochastic process solution of the Langevin equation (6), averaged over the different realizations of the noise (7), are given by derivatives of \mathcal{Z} :

$$C(i_1, t_1, \dots, i_k, t_k) = \frac{\delta^k \mathcal{Z}}{\delta \eta_{i_1}(t_1) \cdots \delta \eta_{i_k}(t_k)}, \quad (89)$$

$$R(i_1, t_1, \dots, i_{k-1}, t_{k-1}; i_k, t_k) = \frac{\delta^k \mathcal{Z}}{\delta \eta_{i_1}(t_1) \cdots \delta \eta_{i_{k-1}}(t_{k-1}) \delta \hat{\eta}_{i_k}(t_k)}, \quad (90)$$

$$R(i_1, t_1, \dots, i_{k-2}, t_{k-2}; i_{k-1}, t_{k-1}, i_k, t_k) = \frac{\delta^k \mathcal{Z}}{\delta \eta_{i_1}(t_1) \cdots \delta \eta_{i_{k-2}}(t_{k-2}) \delta \hat{\eta}_{i_{k-1}}(t_{k-1}) \delta \hat{\eta}_{i_k}(t_k)}, \quad (91)$$

the sources being set to zero after calculating the derivatives. In other words, the correlation and response functions are averages taken with the normalized weight $\exp(-S_{\text{EFF}})$. The response fields $-i\hat{s}_i(t)$ are inserted at the times where a magnetic field h_i has perturbed the system.

\mathcal{Z} can be written in a very compact form if one introduces the super-field formulation of stochastic processes as explained in ref. 41. In this approach one first enlarges (space)-time to include two Grassmann coordinates θ and $\bar{\theta}$, i.e., $t \rightarrow a = (t, \theta, \bar{\theta})$. The dynamic variables $s_i(t)$ and the auxiliary variable $i\hat{s}_i(t)$ together with the fermionic ones $\psi_i(t)$ and $\bar{\psi}_i(t)$ are encoded in a super-field:

$$\Phi_i(a) = s_i(t) + \bar{\theta}\psi_i(t) + \bar{\psi}_i(t)\theta + i\hat{s}_i(t)\theta\bar{\theta}. \quad (92)$$

With these definitions,

$$\mathcal{Z}[\zeta] = \int \mathcal{D}\vec{\Phi} \exp\left(\frac{1}{2} \int da \Phi_i(a) D_a^{(2)} \Phi_i(a) - \int da H[\vec{\Phi}(a)] + \int da \Phi_i(a) \zeta_i(a)\right) \quad (93)$$

with $da \equiv dt d\theta d\bar{\theta}$, $\vec{\Phi} \equiv (\Phi_1, \dots, \Phi_N)$, $\zeta_i(a) \equiv \eta_i(t) \bar{\theta}\theta + \hat{\eta}_i(t)$, and the dynamic operator $D_a^{(2)}$ is defined as

$$-D_a^{(2)} = 2T \frac{\partial^2}{\partial \theta \partial \bar{\theta}} + 2\theta \frac{\partial^2}{\partial \theta \partial t} - \frac{\partial}{\partial t}. \quad (94)$$

If the model is spherically constrained, one can absorb the constraint in the quadratic part of the action by redefining

$$D_a^{(2)} \rightarrow D_a^{(2)} + \mu_s(a) \quad (95)$$

with $\mu_s(a)$ a super-Lagrange multiplier, with the form (92). If one deals, instead, with the soft-spin model, the non-quadratic term $V_I[\Phi]$ appears in H .

The super-symmetric notation allows one to encode in the single super-correlator

$$Q_{ij}(a, b) \equiv \langle \Phi_i(a) \Phi_j(b) \rangle \quad (96)$$

the two-time correlators and responses, $C_{ij}(t, t')$ and $R_{ij}(t, t')$. Indeed, the super-correlator $Q_{ij}(a, b)$ has 16 ‘‘components’’. However, many of them vanish or are related to each other due to the symmetries that should hold in a physical situation,⁽⁴¹⁾ such as ghost number conservation and causality. Imposing these properties, one is left with a simpler expression for Q_{ij} that encodes the physical self-correlation and the linear response:

$$Q_{ij}(a, b) = C_{ij}(t_a, t_b) + (\bar{\theta}_b - \bar{\theta}_a)(\theta_b R_{ij}(t_a, t_b) - \theta_a R_{ji}(t_b, t_a)). \quad (97)$$

Note that the terms in which two fermions appear equal the response functions. The equilibrium properties can also be expressed as super-symmetries and constrain further the two-point super-correlator imposing TTI and FDT. Similarly, all the three-point functions defined in Section 2 are contained in $Q_{ijk}(a, b, c) \equiv \langle \Phi_i(a) \Phi_j(b) \Phi_k(c) \rangle$.

We are interested in correlations and responses averaged over the disorder. The fact that \mathcal{Z} is equal to 1 in the absence of sources, independently of the realization of the disorder,⁽⁴²⁾ implies that these averaged functions are generated by $[\mathcal{Z}]_J$. Since the generating functional itself (and not its logarithm) has to be averaged one can avoid the use of replicas (if one uses generic initial conditions that are not correlated with the random interactions). The only term that depends on disorder in the action in \mathcal{Z} is H_J . The average over the probability distribution of the couplings (84) and (85) reads

$$\begin{aligned} [e^{-\int da H_J[\bar{\Phi}(a)]}]_J &\equiv e^{-NH_{\text{eff}}} \\ &= \prod_{i_1 < \dots < i_p} \left[1 + \frac{\alpha p!}{N^{p-1}} \left(\cosh \left(\tilde{J} \int da \Phi_{i_1}(a) \dots \Phi_{i_p}(a) \right) - 1 \right) \right] \\ &= \exp \left[N\alpha \frac{1}{N^p} \sum_{i_1 \dots i_p} \left(\cosh \left(\tilde{J} \int da \Phi_{i_1}(a) \dots \Phi_{i_p}(a) \right) - 1 \right) \right] \end{aligned} \quad (98)$$

at leading order in N .

The next step to take is to disentangle this effective Hamiltonian and to reduce the full problem to a single-spin one that will be evaluated using saddle-point methods. The manipulation of the dynamic effective Hamiltonian is done following the same steps as in the replica static calculation. For this one takes advantage of the empirical correspondence between super-coordinates and replica indices.⁽⁴³⁾

For fully-connected models, the reduction to a single spin problem can be done by introducing a *global* two-point function $Q(a, b)$ or the *global* Parisi matrix Q_{ab} in replica terms. For dilute models the procedure is much more difficult. One possibility is to introduce the whole set of multi-point super-correlators $Q(a, b)$, $Q(a, b, c)$, etc. This route was followed in the original treatment of the statics of the Viana–Bray model.⁽³⁰⁾ We follow here a different strategy that was introduced by Monasson⁽³⁴⁾ (see also ref. 33) in the replica static context. It is based upon the introduction of a functional order parameter.

Let us define $c(\Psi)$ as the fraction of sites with super-field Φ_i identical to a chosen value Ψ :

$$c(\Psi) \equiv \frac{1}{N} \sum_{i=1}^N \delta[\Psi - \Phi_i], \quad (99)$$

where the functional δ enforces that $\Psi(a) = \Phi_i(a)$ for all values of the super-coordinate a . Note that c is normalised, $\int \mathcal{D}\Psi c(\Psi) = 1$.

As emphasized in ref. 34, the mean-field character of the models implies that their effective Hamiltonian, after performing the average over disorder, can be expressed in terms of such a *global* functional order parameter. For the dilute p spin models under consideration here, one obtains

$$\begin{aligned} -H_{\text{EFF}}[c] = & \alpha \int \mathcal{D}\Psi_1 \cdots \mathcal{D}\Psi_p c(\Psi_1) \cdots c(\Psi_p) \\ & \times \left[\cosh \left(\tilde{J} \int da \Psi_1(a) \cdots \Phi_p(a) \right) - 1 \right]. \end{aligned} \quad (100)$$

We enforce the definition of $c(\Psi)$ in the generating functional by introducing an identity in its path integral representation:

$$\begin{aligned} 1 = & \int \mathcal{D}c \mathcal{D}i\hat{c} \exp \left[\int \mathcal{D}\Psi i\hat{c}(\Psi) \left(Nc(\Psi) - \sum_{i=1}^N \delta[\Psi - \Phi_i] \right) \right] \\ = & \int \mathcal{D}c \mathcal{D}i\hat{c} \exp \left[\int \mathcal{D}\Psi Ni\hat{c}(\Psi) c(\Psi) - \sum_{i=1}^N i\hat{c}(\Phi_i) \right]. \end{aligned} \quad (101)$$

After doing so we achieve our goal of disentangling the N degrees of freedom

$$[\mathcal{Z}]_J = \int \mathcal{D}c \mathcal{D}i\hat{c} \exp[-NG] \quad (102)$$

with

$$G = - \int \mathcal{D}\Psi \ i\hat{c}(\Psi) c(\Psi) + H_{\text{EFF}} - \ln \left[\int \mathcal{D}\Psi \ e^{\frac{1}{2} \int da \Psi(a) D_a^{(2)} \Psi(a) - \int da v(\Psi(a)) - i\hat{c}(\Psi)} \right]. \quad (103)$$

We can now evaluate \mathcal{Z} with the saddle-point method. Enforcing that G is stationary with respect to c and \hat{c} yields

$$c^{\text{SP}}(\Psi) = \lambda \exp \left[\frac{1}{2} \int da \Psi(a) D_a^{(2)} \Psi(a) - \int da v(\Psi(a)) - i\hat{c}^{\text{SP}}(\Psi) \right], \quad (104)$$

$$\hat{c}^{\text{SP}}(\Psi) = \left. \frac{\delta H_{\text{EFF}}[c]}{\delta c(\Psi)} \right|_{c^{\text{SP}}}, \quad (105)$$

with λ a normalization constant.

Let us give the interpretation of such equations. The correlation functions computed with the normalized weight $c^{\text{SP}}(\Psi)$ are the averages of the original single-site functions:

$$\int \mathcal{D}\Psi \ c^{\text{SP}}(\Psi) \Psi(a_1) \cdots \Psi(a_k) = \frac{1}{N} \sum_i [\langle \Phi_i(a_1) \cdots \Phi_i(a_k) \rangle]_J. \quad (106)$$

To enlighten the physical meaning of Eqs. (104) and (105), let us consider the fully-connected limit, with $\alpha \rightarrow \infty$, $\tilde{J} \rightarrow 0$ and $J_0^2 = 2\alpha\tilde{J}^2$ finite. Expanding the hyperbolic cosine in (100), only the first term of the series survives:

$$H_{\text{EFF}} \rightarrow \frac{J_0^2}{4} \int \mathcal{D}\Psi_1 \cdots \mathcal{D}\Psi_p \ c(\Psi_1) \cdots c(\Psi_p) \int da db \Psi_1(a) \Psi_1(b) \cdots \Psi_p(a) \Psi_p(b) \quad (107)$$

The relation between \hat{c}^{SP} and c^{SP} thus becomes

$$\hat{c}^{\text{SP}}(\Psi) = \frac{pJ_0^2}{4} \int da db Q^{\text{SP}}(a, b)^{\bullet(p-1)} \Psi(a) \Psi(b), \quad (108)$$

$$Q^{\text{SP}}(a, b) \equiv \int \mathcal{D}\Psi \ c^{\text{SP}}(\Psi) \Psi(a) \Psi(b)$$

and \bullet represents the direct or Hadamard product (see Appendix A). We thus obtain the SUSY form of the well-known single-spin equation for fully-connected models: averages of Ψ are taken with a Gaussian weight (apart from the soft-spin v term), to be determined self-consistently through Q^{SP} . It is the equation of a single degree of freedom evolving through a Langevin equation with a retarded interaction and a colored noise, which are to be expressed self-consistently in terms of the correlation and the response of the process.⁽⁴⁴⁾

In the dilute case, one can also follow this idea, but the self-consistency equations cannot be written solely in terms of the two-point function. The whole hierarchy of many-coordinate correlations appears:

$$c^{\text{SP}}(\Psi) = \lambda \exp \left[\frac{1}{2} \int da \Psi(a) D_a^{(2)} \Psi(a) - \int da v(\Psi(a)) - \alpha p \sum_{n=1}^{\infty} \frac{\tilde{J}^{2n}}{(2n)!} \right. \\ \left. \times \int da_1 \cdots da_{2n} \Psi(a_1) \cdots \Psi(a_{2n}) Q_{2n}(a_1, \dots, a_{2n})^{\bullet(p-1)} \right], \quad (109)$$

$$Q_{2n}(a_1, \dots, a_{2n}) \equiv \int \mathcal{D}\Psi c^{\text{SP}}(\Psi) \Psi(a_1) \cdots \Psi(a_{2n}). \quad (110)$$

This equation cannot be solved exactly. A set of approximation schemes can be envisaged. In the following we critically discuss some of the approximations one can use.

3.4. Approximation Schemes

In this subsection we discuss some approximations to the saddle-point equations (109) and (110) that are exact in the thermodynamic limit and determine the behavior of all global correlators.

3.4.1. Cutting the Series

The simplest approximation one can envisage is to simply cut the series appearing in the right-hand side of (109) after the first term in such a way that only $Q(a, b)$ enters the approximated equation. This is equivalent to proposing that \hat{c} is quadratic. It is easy to see that with this approximation we recover the fully-connected counterpart model, with $J_0 = \tilde{J} \sqrt{2\alpha}$. The solution does not go beyond the results already known for this case.^(7, 23, 39)

Presumably, better approximations are obtained by progressively keeping some higher order correlations. More precisely, one could cut the series after the second term and then derive (exactly in the spherical case or with a further approximation to treat the effect of the soft-spin term v for

the Ising problem) a set of dynamic equations coupling $Q(a, b)$ and $Q(a, b, c, d)$ only. One could also cut the series after the third term and then derive a set of dynamic equations coupling $Q(a, b)$, $Q(a, b, c, d)$ and $Q(a, b, c, d, e, f)$ and so on and so forth.

This is the kind of approach used when deriving and cutting BBGKY hierarchies in field-theoretical models or condensed-matter problems. It is also similar to the procedure used by Viana and Bray⁽³⁰⁾ in their study of the statics of the dilute \pm spin-glass model with two-body interactions close to its transition temperature where the Q 's (in replica space) can be assumed to be small.

3.4.2. An Iterative Procedure

A different procedure consists in evaluating $c(\Psi)$ (and the super-correlators) iteratively. The idea is:

- (i) In the first step one uses the simplest *Ansatz* for $c(\Psi)$ and evaluates H_{eff} . (We shall discuss two ways of implementing this initial step.)
- (ii) Next, one uses Eq. (104) [after replacing \hat{c} with (105)] to obtain an improved expression for $c^{\text{sp}}(\Psi)$.
- (iii) With this form one calculates H_{eff} again and goes to (ii).

This procedure follows very closely the proposal of Biroli and Monasson to compute the density of states of sparse random matrices.⁽²⁷⁾ The SUSY formulation of the dynamic generating functional makes the treatment of the dynamic problem very similar to the static calculation that uses the replica trick (see ref. 43 for a recent discussion of the relation between SUSY and replica analysis).

An additional feature to be considered in this iteration is how we update the constraint on the normalization of the correlation, i.e., the fact that $Q(a, a) = 1$. For the spherical model we shall demand that this constraint be valid at each step of the iteration. This means that we shall modify the Lagrange multiplier in such a way as to impose the constraint.

Importantly enough, after the first iteration one accesses the functional order parameter $c(\Psi)$ and hence *all* many-point correlations simultaneously though approximately. One can expect that even after using only one step of the iteration this method will yield better information than simply cutting the series keeping only a finite number of terms. This is indeed the case when studying the spectral properties of random matrices.

3.4.3. First Step: The Effective Medium Approximation (EMA)

In the first step we use the simplest approximation that captures the same behavior as the “effective-medium approximation”.^(27, 28) In this

approximation, one treats the environment of each point on the random graph in a uniform manner. In the calculation of the spectral density of sparse random matrices this approximation yields a symmetric distribution with a finite support that qualitatively resembles a semi-circle. Thus, the result is a simple modification with respect to the usual Gaussian case. This approximation fails to capture the effect of highly connected sites that clearly deviate from the effective-medium assumption. These imply the appearance of tails in the density of states that are not obtained at this level of the calculation.

The dynamic effective-medium approximation can be done in at least two ways:

Cutting the Series

Going back to what we discussed above, one can use as the starting $c(\Psi)$ the result of cutting the series in (109) after the first term, deriving and solving a self-consistent equation for $Q(a, b)$ and using this as an input to compute $c(\Psi)$.

Gaussian Approximation

A slightly different (but qualitatively equivalent) starting point is to propose a Gaussian *Ansatz* for $c(\Psi)$:

$$c_{\text{EMA}}(\Psi) = (\det Q)^{-1/2} \exp \left(-\frac{1}{2} \int da db \Psi(a) Q^{-1}(a, b) \Psi(b) \right). \quad (111)$$

One can easily check that the denominator ensures the normalization of $c(\Psi)$ and that $Q(a, b)$ is correctly given by

$$Q(a, b) = \frac{1}{\sqrt{\det Q}} \int \mathcal{D}\Psi \Psi(a) \Psi(b) \times \exp \left(-\frac{1}{2} \int da' db' \Psi(a') Q^{-1}(a', b') \Psi(b') \right).$$

This form is closer to the approximation used in refs. 27 and 28.

To advance further it is convenient to express the c -dependent effective action in terms of $Q(a, b)$. This can be done exactly for the spherical problem. When we deal with the soft-spin case, instead, the non-quadratic term $V_I(\Psi)$ has to be treated within an additional approximation. We shall discuss this point later. For the moment we focus on the spherical models. After rather simple manipulations one finds

$$[\mathcal{Z}]_J = \int \mathcal{D}Q e^{-S_{\text{EFF}}(Q)}, \quad \text{with}$$

$$2S_{\text{EFF}}^{\text{SPH}}(Q) = \text{Tr} \ln Q + \int da db \delta(a-b)(D_a^{(2)} + \mu_s(a)) Q(a, b) - 2H_{\text{EFF}}^J(Q).$$

The superscript J in H_{EFF} indicates that the form of the effective Hamiltonian is dictated by the disorder-dependent term H_J . Its variation with respect to Q yields

$$\frac{\delta S_{\text{EFF}}^{\text{SPH}}(Q)}{\delta Q} = 0 = \frac{1}{2} Q^{-1}(a, b) + \frac{1}{2} (D_a^{(2)} + \mu_s(a)) \delta(a, b) - \frac{\delta H_{\text{EFF}}^J(Q)}{\delta Q(a, b)}. \quad (112)$$

Multiplying this equation operationally by $Q(b, a')$ (see Appendix A) the dynamic equation takes the more familiar Schwinger–Dyson form

$$(D_a^{(2)} + \mu_s(a)) Q(a, b) + \delta(a-b) + \int da' \Sigma(a, a') Q(a', b) = 0, \quad (113)$$

with

$$\Sigma(a, a') = -2 \frac{\delta H_{\text{EFF}}(Q)}{\delta Q(a, a')}. \quad (114)$$

As mentioned above, we require that Q satisfy the spherical constraint. Thus, we fix the evolution of $\mu_s(a)$ to ensure the validity of this constraint.

It is now instructive to see how one recovers the well-known equations for the fully-connected model. We have seen in Eq. (107) how the effective Hamiltonian simplifies in this limit. Plugging in the Gaussian *Ansatz* (111), H_{EFF}^J becomes a simple function of $Q(a, b)$,

$$H_{\text{EFF}}^J(Q) = \frac{J_0^2}{4} \int da db Q^p(a, b), \quad (115)$$

and the self-energy $\Sigma(a, b)$ is easily deduced from this expression. A way to prove that the Gaussian *Ansatz* (111) is exact for the fully-connected case is to check that the *exact* equation for $c(\Psi)$ coincides with the one obtained from the Gaussian *Ansatz* and the saddle-point evaluation.

In the dilute case, one has to distinguish between the cases $p = 2$ and $p \geq 3$. Indeed, consider the expression (100), where one uses the Gaussian *Ansatz* (111). The integrals over Ψ can be formally performed with an expansion of the hyperbolic cosine and the use of Wick’s theorem. When $p = 2$, all the terms can be written explicitly and the asymptotic behavior of

the two-point correlator can be worked out. We do not reproduce the calculations here as they do not give more insight into the physics of the problem: as expected, at this level of the approximation one recovers a behavior typical of the fully-connected counterpart model, as was shown in a more direct way in ref. 16. For $p \geq 3$, H_{EFF}^J can only be expressed as a series expansion involving the super-correlators. The properties of the Hadamard and operational products explained in the Appendix allow one to prove that the solution to this equation has the same properties as the one for the fully-connected $p = 3$ model.

Even if the EMA equations found with the Gaussian *Ansatz* are much more complicated than the one obtained by cutting the series, one can check that they contain the same qualitative information, which is that of the fully-connected counterpart model.

Soft-Spin Models

The term $v_I(\Psi) = a(\Psi^2 - 1)^2$ in H is not quadratic and one has to resort to an additional approximation to treat it. One possibility, for models with continuous dynamic transitions, is to identify the simplest non-trivial term that it generates in a series expansion around the critical temperature at which one expects the order parameter to vanish. This is the proposal in ref. 45, later used in the study of the dynamics of the Sherrington–Kirkpatrick model in ref. 23. Another possibility, which can also be applied to models with discontinuous dynamic transitions, was used in ref. 46 and corresponds to treating this term in the mode-coupling approximation. The terms obtained with the first and the second procedure are slightly different. Even if they modify the details of the dynamic solution in the fully-connected case, they do not modify the qualitative features of it. We expect these same generic features also in the dilute case.

3.4.4. Second Step: The Single Defect Approximation (SDA)

In the study of the density of states of sparse random matrices one can go beyond the Gaussian approximation.^(27–29) The same kind of approach can be used in the analysis of the dynamics of dilute disordered models. The idea is to use Eqs. (104) and (105) iteratively. In the first step one replaces $c_{\text{EMA}}(\Psi)$ on the right-hand side of (105) and computes its improved functional form. In the case of matrices this gives access to the tails in the eigenvalue distribution. We expect that this improved approximation will also modify the result for the dynamic behavior considerably since it will capture, at least partially, the effect of heterogeneities in the connectivity of the sites in the random (hyper) graph.

In order to clarify the way in which the iteration is implemented, let us neglect the soft-spin term (this, of course, is not correct if we are treating

an Ising spin problem, in which case, as already said, we have to treat v with a further approximation). Thus, $H_{\text{EFF}} = H_{\text{EFF}}^J$ and in order to compute the first iteration we insert

$$\frac{\delta H_{\text{EFF}}^J}{\delta c(\Psi_1)} = -\alpha p + \alpha p \int \mathcal{D}\Psi_2 \cdots \mathcal{D}\Psi_p c(\Psi_2) \cdots c(\Psi_p) \times \cosh \left(\tilde{J} \int da \Psi_1(a) \cdots \Psi_p(a) \right) \quad (116)$$

in the right-hand side of Eq. (104) and expand the last term as

$$c_{\text{SDA}}(\Psi_1) = \lambda' e^{\frac{1}{2} \int da \Psi_1(a) [D_a^{(2)} + \mu_s(a)] \Psi_1(a)} \sum_{k=0}^{\infty} \frac{e^{-\alpha p} (\alpha p)^k}{k!} \times \left[\int \mathcal{D}\Psi_2 \cdots \mathcal{D}\Psi_p c_{\text{EMA}}(\Psi_2) \cdots c_{\text{EMA}}(\Psi_p) \times \cosh \left(\tilde{J} \int da \Psi_1(a) \cdots \Psi_p(a) \right) \right]^{*k}. \quad (117)$$

For the sake of clarity we concentrate on the case $p = 2$. Using now the Gaussian expression for $c_{\text{EMA}}(\Psi)$ with variance Q_{EMA} , we find

$$c_{\text{SDA}}(\Psi) = \lambda' \sum_{k=0}^{\infty} \frac{e^{-2\alpha} (2\alpha)^k}{k!} \exp \left[-\frac{1}{2} \int da db \Psi(a) A_k(a, b) \Psi(b) \right] \quad (118)$$

with

$$A_k(a, b) \equiv -\delta(a-b) [D_a^{(2)} + \mu_s(a)] - k \tilde{J}^2 Q_{\text{EMA}}(a, b). \quad (119)$$

Rewriting this equation in the form

$$\delta(a, b) \equiv [D_a^{(2)} + \mu_s(a)] A_k^{-1}(a, b) - k \tilde{J}^2 \int db' Q_{\text{EMA}}(a, b') A_k^{-1}(b', b), \quad (120)$$

we realize that this is a Schwinger–Dyson equation for a “system” (the unknown A_k^{-1}) in the presence of a complex external bath: a white bath giving rise to the dynamic operator $D_a^{(2)}$ and a “colored bath” function of the SUSY self-energy Q_{EMA} .^(5, 47) The latter has a slow relaxation, of glassy type, since in the EMA the dilute model behaves in a very similar manner to its fully-connected relative. This means that the SUSY correlator Q_{EMA} encodes a correlation and a response with two two-time regimes, a stationary and an aging one, the former controlled by the temperature of the

external bath and the latter characterized by an effective temperature that differs from that of the environment. If we assume—a hypothesis that has to be checked!—that the effect of this colored bath on the system is as discussed in refs. 5 and 47, the latter will follow the dynamics dictated by the complex bath with two time-scales and two values of the effective temperature.

The spirit of the iteration is to use the previously determined value of $Q(a, b)$ as an input. For the spherical model $\mu_s(t)$ is also a function that is self-consistently determined to impose the spherical constraint at equal times. We believe that a convenient way to deal with the constraint is to impose it on each step of the approximation. Thus, we let the value of $\mu_s(t)$ appearing in the SDA level of the calculation free and we fix it by imposing that the correlations at equal times be normalized to one. To this end, it is convenient to impose the spherical condition on A_k^{-1} since this ensures, automatically, the normalization of Q_{SDA} at equal times with $\lambda' = 1$.

Now, using $Q_{\text{SDA}}(a, b) \equiv \int \mathcal{D}\Psi \Psi(a) \Psi(b) c_{\text{SDA}}(\Psi)$, we have

$$Q_{\text{SDA}}(a, b) = \lambda' \sum_{k=0}^{\infty} \frac{e^{-2\alpha}(2\alpha)^k}{k!} \frac{A_k^{-1}(a, b)}{\sqrt{\det A_k}}. \quad (121)$$

One recognizes in (121) a series with the same structure as Eq. (29) in ref. 28. As explained in this reference, this series captures the effect of the heterogeneities in the connectivity of sites in the random graph. Each term represents the contribution of sites connected to k neighbors. The SDA super-correlator is then a very different entity from the EMA one. It is given by the “superposition” of independent aging systems (labeled by k) each weighted with the Poissonian distribution.

The complete solution of the problem, at this step, needs more work. We should be able to estimate the behavior of this two-time dependent series but this is not an easy task. It would be very interesting to check how the results in ref. 16 are recovered with this approach. In particular, a property of this solution that is not easily seen from the calculations above is the fact that the asymptotic value of the Lagrange multiplier actually diverges when $p = 2$.

4. CONCLUSIONS

In this paper we discussed several aspects of the dynamics of disordered spin models.

The first part is general and deals with the generic properties of many-point functions in equilibrium and the modifications expected out of equilibrium for models with slow dynamics. In particular we formulate an

explicit conjecture, Eq. (81), for the out-of-equilibrium FD relations relating the multi-time correlation and response functions. This conjecture proves to be correct for Gaussian aging processes and its consequences should be amenable to numerical checks; for instance Eq. (82).

In the second part of the paper we presented an approach to the disorder-averaged dynamics of dilute random spin models. Since with this approach we study the dynamic generating functional averaged over disorder we only have access to typical properties and we cannot follow the single spin dynamics in the particular background of a quench-disorder realization. This should be contrasted with the recent advances in static calculations which allow a detailed sample-by-sample analysis of similar models.^(13, 37)

When making contact between the generic method we discussed here and the explicit solution to the spherical dilute spin-glass model with two-body interactions⁽¹⁶⁾ we see that with each level of the approximation we access different time-scales. Indeed, with the first EMA we only see the dynamics in a first long-times scale that resembles strongly the fully-connected partner of the model. With this calculation the fluctuations in the connectivities are averaged out and the result is consistent with it. One step beyond one sees the fluctuations in the number of first neighbors of the sites and the disorder-averaged dynamics feels the existence of special sites with larger connectivity than the average one, in a longer (still aging) time-scale. It would be interesting to understand how the existence and number of these time-scales and/or the behavior of the system in them is modified when including more levels in the iteration.

APPENDIX A: PROPERTIES OF SUPER-CORRELATORS

Two products between SUSY correlators can be defined. These are the convolution and the Hadamard product:

$$Q^{(1)}(a, b) \otimes Q^{(2)}(b, c) \equiv \int db Q^{(1)}(a, b) Q^{(2)}(b, c), \quad (122)$$

$$Q^{(1)}(a, b) \bullet Q^{(2)}(a, b) \equiv Q^{(1)}(a, b) Q^{(2)}(a, b).$$

These products are associative and commutative.

If one multiplies two SUSY correlators of the form (97) with either of these products one obtains another SUSY correlator of the same form. Its new ‘‘components’’ are given by functionals of the components of the original SUSY correlators. Indeed,

$$Q(a, b) = (Q^{(1)} \otimes Q^{(2)})(a, b) = C(t_a, t_b) + (\bar{\theta}_b - \bar{\theta}_a)(\theta_b R(t_a; t_b) - \theta_a R(t_2; t_1)) \quad (123)$$

with

$$C(t_a, t_b) = \int dt_c [R^{(1)}(t_a, t_c) C^{(2)}(t_c, t_b) + C^{(1)}(t_a, t_c) R^{(2)}(t_b, t_c)], \quad (124)$$

$$R(t_a; t_b) = \int dt_c R^{(1)}(t_a, t_c) R^{(2)}(t_c, t_b),$$

and

$$Q(a, b) = (Q^{(1)} \bullet Q^{(2)})(a, b) = C(t_a, t_b) + (\bar{\theta}_b - \bar{\theta}_a)(\theta_b R(t_a; t_b) - \theta_a R(t_2; t_1)) \quad (125)$$

with

$$C(t_a, t_b) = C^{(1)}(t_a, t_b) C^{(2)}(t_a, t_b), \quad (126)$$

$$R(t_a; t_b) = R^{(1)}(t_a, t_b) C^{(2)}(t_a, t_b) + C^{(1)}(t_a, t_b) R^{(2)}(t_a, t_b).$$

It is very simple to check that time-translation invariance and the fluctuation–dissipation relation are conserved by these products. More precisely, if one multiplies two super-correlators satisfying these properties, the results will also verify them. Moreover, multiplying two super-correlators evaluated in times that are sufficiently far away such that they evolve in their slow (aging) scale, the result will also be slow. This property is apparent for the Hadamard product. To prove it for the convolution one needs to separate the integration time into several intervals and approximate the integrals as is usually done when solving the dynamics of fully-connected models (see, e.g., ref. 5). By induction one can then prove that any k th power in the Hadamard or convolution sense of a SUSY correlator in the fast, stationary regime yields a result within this regime. Similarly, any Hadamard or convolution power of SUSY correlators in the slow, aging regime stays in the same regime. This property is very useful for solving the EMA equations.

ACKNOWLEDGMENTS

L.F.C. is a research associate at ICTP Trieste and a Fellow of the Guggenheim Foundation. We acknowledge financial support from the ACI Jeunes Chercheurs “Algorithmes d’optimisation et systèmes désordonnés quantiques.” This research was supported in part by the National Science Foundation under Grant No. PHY99-07949.

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